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Uncertainty Principle Limits on the Cosmological Constant

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The identification of the cosmological constant term with the energy density of the vacuum enables lower and upper limits to be placed on its value. The upper limit arises from the constraint that the total zero-point energy, which also gravitates, should not dominate cosmological dynamics, while the lower limit can result from the operational requirement that the vacuum-energy shifts over atomic or nuclear scales be at least measurable over Hubble time scales.

The cosmological constant term introduced and discarded by Einstein has been recently rejuvenated as representing the energy-momentum of the vacuum in quantum field theory (e.g., Zeldovich and Novikov, 1971). Since for a covariant definition the vacuum should not single out any preferred frame, its energy-momentum tensor $(T_{\mu\nu})_{\text{vac}}$ must be uniquely of the form $(T_{\mu\nu})_{\text{vac}} = \rho_{\text{vac}} g_{\mu\nu}$, where ρ_{vac} is the vacuum energy density (The uniqueness of $g_{\mu\nu}$ arises, as is well known, from the fact that if there is no preferred frame, there are no preferred vectors, which is possible only if all vectors are eigenvectors, i.e., if $X_\mu^\nu V_\nu = KV_\mu$ for all K; then X_μ^ν is proportional to g_μ^ν). Thus the Einstein field equations with the vacuum term take the form

$$G^{\mu\nu} = 8\pi G(T^{\mu\nu} + T^{\mu\nu}_{(\text{vac})}) = 8\pi G(T^{\mu\nu} + g^{\mu\nu}\rho_{\text{vac}})$$

Comparing with the field equations with a cosmological $\Lambda g_{\mu\nu}$ term, we have the identification

$$\Lambda = \frac{8\pi G}{c^4}\rho_{\text{vac}} = \kappa\rho_{\text{vac}} \tag{1}$$

[see Sivaram et al. (1976) for a more general treatment]. Now the energy of the vacuum or zero-point energy due to quantum fluctuations of various fields would in general relativity also contribute to the gravitational field and hence to cosmological dynamics.

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Zeldovich has suggested a possible physical picture to estimate ρ_{vac} and hence Λ. One can assume the vacuum to consist of particles of mass m at a separation of $\lambda \sim \hbar/mc$ (i.e., a particle in a Compton volume). Then the energy density would be $\sim m(mc/\hbar)^3 \sim m^4c^3/\hbar^3$. For electrons and protons this would give absurdly large values of $\sim 10^5$ and $\sim 10^{17}$ g/cm^3, respectively. Only particles with $m \sim 10^{-3}$ eV [i.e., $\sim 10^{-9} m$ (electron)] would give reasonable vacuum densities if they uniformly fill phase space. Again following Zeldovich, we can consider the vacuum energy as arising from gravitational interactions of energy $\sim Gm^2/\lambda$ of nearby particles, which would imply a mean vacuum mass density of $(Gm^2/\lambda)/c^2\lambda^3 \approx Gm^6c^2/\hbar^4$. These suggestions are, however, rather ad hoc, since on dimensional grounds ρ_{vac} can be of the form $(m^4c^3/\hbar^3)(Gm^2/\hbar c)^n$ with n any number. We shall consider below alternative ways of estimating Λ from more stringent constraints.

It may be remarked that our arguments pertain to Λ in the present epoch of the universe and not the large vacuum energy densities at the very early epochs, which have been invoked recently, involving GUT phase transitions in inflationary models. Our limits would apply to any residual vacuum energies that may be present at the current epoch and the corresponding Λ. The presence of such a term can help ameliorate problems with the age of the universe (i.e., discrepancy between the Hubble age and the age of the oldest objects such as globular clusters) that arise in inflationary models which require closure density, implying vast amounts of dark matter (Sivaram, 1985).

The zero-point energy due to the curvature of space-time is identified by Sakharov (1968) with a term identical to the Hilbert action of Einstein's theory, i.e., the Lagrange density of the zero-point energy is expanded as

$$L(R) \approx \text{const} \times \hbar c R \int k\, dk + \text{higher order terms} \tag{2}$$

where k is the wave number. The wave number is assumed cut off at the Planck length, for which $k_{\text{Pl}} \approx (c^3/\hbar G)^{1/2} \approx 10^{33}$ cm^{-1}. If we assume that the average curvature of space corresponds to that of Λ and introduce the cutoff in the integral at k_{Pl}, then we can write the zero-point energy density as

$$\rho_{\text{vac}} \approx \tfrac{1}{2} \hbar c \Lambda k_{\text{Pl}}^2 \tag{3}$$

This energy density should not exceed the closure density ρ_c, which for an observed Hubble's constant H_0 is given as $\rho_c \approx 3H_0^2c^2/8\pi G$. From the equality $\rho_{\text{vac}} \approx \rho_c$, we get the upper limit for Λ as

$$\Lambda \leq \frac{6H_0^2 c}{8\pi G\hbar k_{\text{Pl}}^2} \tag{4}$$

which, substituting for $H_0 \approx 100$ km |s| Mpc and k_{Pl}^2, etc., would give $\Lambda \leq 10^{-57}\ \text{cm}^{-2}$.

For the lower limit, to be arrived at from microphysical considerations, we would impose the operational constraint that for the vacuum-energy shifts over atomic or nuclear scales to be physically meaningful they should be at least measurable over Hubble time scales. This operational requirement was invoked in a somewhat different context in earlier papers (e.g., Sivaram, 1982a). It was pointed out (Sivaram, 1982b, 1984) that the characteristic length $e^2/2m_ec^2 = g^2/2m_pc^2 = \hbar/m_\pi c$ was a very pertinent scale for nuclear and atomic fundamental processes (m_π is the pion mass, and e^2 and g^2 relate to electromagnetic and nuclear couplings, and m_e and m_p are electron and proton masses, respectively).

Using equations (1), we obtain the time-energy uncertainty principle in this context in the form (see also Sivaram, 1982a)

$$\frac{\Lambda}{k}\left(\frac{e^2}{m_ec^2}\right)^3 \frac{4\pi}{3}\frac{1}{H_0} \geq \hbar \tag{5}$$

which imposes the lower limit

$$\Lambda \geq 6\hbar H_0 m_e^3 c^2 G/e^6 \tag{6}$$

again giving $\Lambda \geq 10^{-57}\ \text{cm}^{-2}$.

Thus the two limits seem to converge on fixing Λ as $10^{-57}\ \text{cm}^{-2}$! These limits on Λ are much less ad hoc and better defined than earlier attempts to relate it to quantum parameters, and if Λ is to be related to quantum vacuum energy, these are the types of relations one would expect.

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